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Volume Integrals Associated With the Inhomogeneous Helmholtz Equation

II - Cylindrical Region; Rectangular Region

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I. INTEGRATION OVER A FINITE CYLINDRICAL REGION, Fig. 1

The integrals, Eqs. (12,13) [1], see also [2], are of either one of the following forms:

$$\phi^o = \iiint_\Omega (x')^p (y')^q (z')^s dv' \tag{1}$$

$$\phi^s = \iiint_\Omega \rho(x',y',z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^{k} \partial x'^{n-\ell-k}} \left\{\frac{\sin \alpha r'}{r'}\right\} dv' \tag{2}$$

$$\phi^c = \iiint_\Omega \rho(x',y',z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^{k} \partial x'^{n-\ell-k}} \left\{\frac{\cos \alpha \gamma'}{\gamma'}\right\} dv' \tag{3}$$

Letting $x'=x'$, $y'=\zeta\cos\theta$, $z'=\zeta\sin\theta$ and $dv'=dx'dy'dz'=\zeta d\zeta d\theta dx'$, these integrals can be further evaluated as follows: [3]

(a)

$$\phi^o = \iiint_\Omega (x')^p (y')^q (z')^s dv'$$

$$= \frac{(q-1)!!(s-1)!!(2\pi)}{(q+s)!!} \left(\frac{a^{2+q+s}}{2+q+s}\right)\left(\frac{2}{1+p}\ell^{1+p}\right) \quad \text{if } p,q,s \text{ all even} \tag{4a}$$

$$= 0 \quad \text{if any one of } p,q,s \text{ is odd.} \tag{4b}$$

where according to the definition of factorial,

$$(q-1)!! = \frac{(q-1)!}{2^{(\frac{q}{2}-1)}(\frac{q}{2}-1)!} = 1\cdot3\cdot5\ldots(q-1), \quad q \text{ even} \tag{5a}$$

$$(q+s)!! = 2^{(\frac{q+s}{2})}\left(\frac{q+s}{2}\right)! = 2\cdot4\cdot6\ldots s(s+2)\ldots(s+q), \quad q,s \text{ even} \tag{5b}$$

and a is the radius of the cylinder, ℓ the length.

(b) $n=0$, ϕ^s:

$$\phi^s = \iiint_\Omega (x')^p (y')^q (z')^s \frac{\sin\alpha\gamma'}{\gamma'} dv'$$

$$= \iiint_\Omega (x')^p (y')^q (z')^s \sum_{m=o}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} (r')^{2m} dv'$$

$$= \sum_{m=o}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} S_{m,p} \tag{6}$$

where

$$S_{m,p} = \iiint_{\Omega} (x')^p (y')^q (z')^s (r')^{2m} dv' \tag{7}$$

Using the multinomial formula, [4]

$$(r')^{2m} = (x'^2+y'^2+z'^2)^m = \sum_{m_1,m_2,m_3} \frac{m!}{m_1!m_2!m_3!} (x')^{2m_1} (y')^{2m_2} (z')^{2m_3} \tag{8}$$

the integral

$$S_{m,p} = \sum_{m_1,m_2,m_3} \frac{m!}{m_1!m_2!m_3!} \iiint_{\Omega} (x')^{2m_1+p} (y')^{2m_2+q} (z')^{2m_3+s} dv'$$

$$= \sum_{m_1,m_2,m_3} \frac{m!}{m_1!m_2!m_3!} \frac{(\mu-1)!!(\nu-1)!!(2\pi)}{(\mu+\nu)!!} \left(\frac{a^{2+\mu+\nu}}{2+\mu+\nu}\right)\left(\frac{2\ell^{1+\lambda}}{1+\lambda}\right)$$

$$m=0,1,2,\ldots\ , \quad \text{if } \lambda,\mu,\nu \text{ all even}$$

$$= 0 \qquad \text{if any one of } \lambda,\mu,\nu \text{ is odd} \tag{9}$$

where $(\mu-1)!!$, $(\nu-1)!!$, $(\mu+\nu)!!$ are defined as the same as (5a) and (5b),

and,

$$\begin{aligned} \lambda &= 2m_1+p \\ \mu &= 2m_2+q \\ \nu &= 2m_3+s \\ m &= m_1+m_2+m_3 \end{aligned} \tag{10}$$

(c) $n=0$, ϕ^c:

$$\phi^c = \iiint_{\Omega} (x')^p (y')^q (x')^s \frac{\cos\alpha r'}{r'} dv'$$

$$= \iiint_{\Omega} (x')^p (y')^q (x')^s \sum_{m=o}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \frac{(r')^{2m}}{r'} dv'$$

$$= \sum_{m=o}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} C_{m,p} \tag{11}$$

where

$$C_{m,p} = \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{(r')^{2m}}{r'} \, dv'$$

$$= \sum_{m_1,m_2,m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'} \tag{12}$$

$$\lambda = 2m_1+p$$

$$\mu = 2m_2+q$$

$$\nu = 2m_3+s$$

$$m_1+m_2+m_3 = m \tag{13}$$

The integral on the right hand side of Eq.(12) can be evaluated as follows:

$$\iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'} =$$

$$= \frac{(\mu-1)!!(\nu-1)!!}{(\mu+\nu)!!}(2\pi)\cdot I \qquad \text{if } \mu,\nu \text{ all even} \tag{14a}$$

$$= 0 \qquad \text{if any one of } \mu,\nu \text{ is odd.} \tag{14b}$$

In the above expression

$$I = \frac{1}{k+1} \sum_{t=1}^{\frac{1}{2}(k-1)} (-1)^{t+1} \frac{(k+1)(k-1)\ldots(k-2t+3)}{k(k-2)\ldots(k-2t+2)} (a)^{k-2t+1} \int_{-\ell}^{\ell} (x')^{\lambda+2t-2} (a^2+x'^2)^{1/2} dx'$$

$$+ (-1)^{\frac{1}{2}(k-1)} \frac{(k-1)!!}{k!!} \int_{-\ell}^{\ell} (x'^{\lambda+k-1}(a^2+x'^2)^{1/2} - x'^{\lambda+k}) dx' \tag{15}$$

where

$$k = 1+\mu+\nu \tag{16}$$

is odd, and

$$(k-1)!! = 2^{(\frac{k-1}{2})} \cdot (\frac{k-1}{2})! = 2\cdot 4\ldots(k-1) \tag{17a}$$

$$(k)!! = \frac{k!}{2^{(\frac{k-1}{2})}(\frac{k-1}{2})!} = 1\cdot 3\ldots(k) \tag{17b}$$

The integrals

$$\int_{-\ell}^{\ell}(x')^{\lambda+2t-2}(a^2+x'^2)^{1/2}dx'$$

$$= \frac{2}{k_1+1}\sum_{t_1=1}^{\frac{k_1}{2}}(-1)^{t_1+1}\frac{(k\ +1)(k_1-1)\ldots(k_1-2t_1+3)}{(k_1+2)k_1\ldots(k_1-2t_1+4)}(a)^{2t_1-2}\cdot(\ell)^{k_1-2t_1+1}\cdot\ (a^2+\ell^2)^{3/2}$$

$$+\ (-1)^{\frac{k_1}{2}}\frac{(k_1-1)!!}{(k_1+2)(k_1)!!}\,2a^{k_1}(\ell\sqrt{a^2+\ell^2}+a^2 sh^{-1}\frac{\ell}{a})\qquad \text{if } \lambda,\mu,\nu \text{ are even}$$

$$= 0\qquad \text{if any one of } \lambda,\mu,\nu \text{ is odd.}\tag{18}$$

where

$$k_1 = \lambda + 2t - 2\tag{19}$$

is even, and

$$(k_1-1)!! = \frac{(k_1-1)!}{2^{(\frac{k_1}{2}-1)}\cdot(\frac{k_1}{2}-1)!} = 1\cdot3\cdot5\ldots(k_1-1)\tag{20a}$$

$$(k_1)!! = 2^{\frac{k_1}{2}}\cdot(\frac{k_1}{2})! = 2\cdot4\ldots(k_1)\qquad k_1 \text{ is even}\tag{20b}$$

The integral

$$\int_{-\ell}^{\ell}(x')^{\lambda+k-1}(a^2+x'^2)^{1/2}dx'$$

$$= \int_{-\ell}^{\ell}(x')^{k_2}(a^2+x'^2)^{1/2}dx'$$

$$= \frac{2}{k_2+1}\sum_{t_2=1}^{\frac{k_2}{2}}(-1)^{t_2+1}\frac{(k_2+1)(k_2-1)\ldots(k_2-2t_2+3)}{(k_2+2)k_2\ldots(k_2-2t_2+4)}(a)^{2t_2-2}\cdot(\ell)^{k_2-2t_2+1}\cdot\ (a^2+\ell^2)^{3/2}$$

$$= (-1)^{\frac{k_2}{2}}\frac{(k_2-1)!!}{(k_2+2)\cdot(k_2)!!}\cdot 2a^{k_2}(\ell\sqrt{a^2+\ell^2}+a^2 sh^{-1}\frac{\ell}{a})\qquad \text{if } \lambda,\mu,\nu \text{ are even}\tag{21a}$$

$$= 0\qquad \text{if any one of } \lambda,\mu,\nu \text{ is odd.}\tag{21b}$$

where

$$k_2 = \lambda + k - 1 = \lambda + \mu + \nu \tag{22}$$

The integral

$$\int_{-\ell}^{\ell} (x')^{\lambda+k} dx'$$

$$= \frac{2\ell^{\lambda+k+1}}{\lambda+k+1} \qquad \text{if } \lambda,\mu,\nu \text{ are even} \tag{23a}$$

$$= 0 \qquad \text{if any one of } \lambda,\mu,\nu \text{ is odd.} \tag{23b}$$

II. INTEGRATION OVER A RECTANGULAR PARALLELEPIPED, Fig. 2

In the case of a rectangular parallelepiped, the integrals in (14), (15), (16) can be evaluated as follows:

(a)

$$\phi^{o} = \iiint_{\Omega} (x')^{p} (y')^{q} (z')^{s} dv'$$

$$= \frac{8}{(p+1)(q+1)(s+1)} (a)^{q+1} (b)^{s+1} (\ell)^{p+1} \quad \text{if } p,q,s \text{ all are even} \qquad (24a)$$

$$= 0 \qquad \text{if any one of } p,q,s \text{ is odd.} \qquad (24b)$$

where ℓ,a,b is length of the rectangular parallelepiped toward x',y',z'-direction, respectively.

(b) n=0, ϕ^{s}

$$\phi^{s} = \iiint_{\Omega} (x')^{p} (y')^{q} (z')^{s} \frac{\sin \alpha r'}{r'} dv'$$

$$= \iiint_{\Omega} (x')^{p} (y')^{q} (z')^{s} \sum_{m=o}^{\infty} (-1)^{m} \frac{\alpha^{2m+1}}{(2m+1)!} (r)^{2m}$$

$$= \sum_{m=o}^{\infty} (-1)^{m} \frac{\alpha^{2m+1}}{(2m+1)!} S_{m,p} \qquad (25)$$

and

$$S_{m,p} = \iiint_{\Omega} (x')^{p} (y')^{q} (z')^{s} (r')^{2m} dv'$$

$$= \sum_{m_1,m_2,m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{2m_1+p} (y')^{2m_2+\ell} (z')^{2m_3+s} dv'$$

$$= \sum_{m_1,m_2,m_3} \frac{8m!}{m_1! m_2! m_3!} \frac{(a)^{\mu+1} (b)^{\nu+1} (\ell)^{\lambda+1}}{(\lambda+1)(\mu+1)(\nu+1)}$$

$$m=0,1,2,\ldots \qquad \text{if } \lambda,\mu,\nu \text{ all are even} \qquad (26a)$$

$$= 0 \qquad \text{if any one of } \lambda,\mu,\nu \text{ is odd.} \qquad (26b)$$

where

$$m = m_1+m_2+m_3$$

$$\lambda = 2m_1+p$$

$$\mu = 2m_2+q$$

$$\nu = 2m_3+s \tag{27}$$

(c) $n=0$, ϕ^c.

$$\phi^c = \iiint_\Omega (x')^p (y')^q (z')^s \frac{\cos\alpha r'}{r'} dv'$$

$$= \iiint_\Omega (x')^p (y')^q (z')^s \sum_{m=o}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \frac{(r')^{2m}}{r'}$$

$$= \sum_{m=o}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} C_{m,p} \tag{28}$$

and

$$C_{m,p} = \sum_{m_1,m_2,m_3} \frac{m!}{m_1!m_2!m_3!} \iiint_\Omega (x')^\lambda (y')^\mu (z')^\nu \frac{dv'}{r'}$$

$$= \sum_{m_1,m_2,m_3} \frac{m!}{m_1!m_2!m_3!} \int_{-\ell}^{\ell} (x')^\lambda [\int_{-a}^{a} (y')^\mu (\int_{-b}^{b} \frac{(z')}{\sqrt{\alpha^2+z^2}} dz') dy'] dx' \tag{29}$$

where

$$\lambda = 2m_1+p$$

$$\mu = 2m_2+q$$

$$\nu = 2m_3+s$$

$$\alpha^2 = x'^2+y'^2$$

$$m_1+m_2+m_3 = m \tag{30}$$

The integrals in (34) can be evaluated as follows:

(i) $$\int_{-b}^{b} \frac{(z')^\nu}{\sqrt{\alpha^2+z'^2}} dz'$$

$$= \Pi_\nu \; (b)^{\nu-2t+1} (x'^2+y'^2) \sqrt{b^2+x'^2+y'^2} + (-1)^{\nu/2} \frac{2(\nu-1)!!}{(\nu)!!} (x'^2+y'^2)^{\nu/2} \mathrm{sh}^{-1} \frac{b}{\sqrt{x'^2+y'^2}}$$

if ν is even (31a)

$=0$ if ν is odd. (31b)

where Π_ν is defined as an operator as follows:

$$\Pi_\nu = \frac{2}{\nu+1} \sum_{t=1}^{\nu/2} (-1)^{t+1} \frac{(\nu+1)(\nu-1)\ldots(\nu-2t+3)}{\nu(\nu-2)(\nu-4)\ldots(\nu-2t+2)} \tag{32a}$$

$$(\nu-1)!! = \frac{(\nu-1)!}{2^{(\frac{\nu}{2}-1)}(\frac{\nu}{2}-1)!} = 1\cdot 3\ldots(\nu-1) \tag{32b}$$

$$(\nu)!! = 2^{\frac{\nu}{2}}(\frac{\nu}{2})! = 2\cdot 4\ldots\nu \quad , \nu \text{ is even} \tag{32c}$$

(ii)

$$\int_{-a}^{a}\int_{-b}^{b} (y')^\mu (z')^\nu \frac{dy'dz'}{\sqrt{x'^2+y'^2+z'^2}}$$

$$= \Pi_\nu \sum_{n_1,n_2} (-1)^{\nu/2} \frac{n!}{n_1!n_2!}(b)^{\nu-2t+1}(x')^{2n_1} I_1(x')$$

$$+ \sum_{i_1,i_2} (-1)^{\nu/2} \frac{i!}{i_1!i_2!} \frac{2(\nu-1)!!}{(\nu)!!}(b)(x')^{2i_1} I_2(x')$$

$$- \sum_{i_1,i_2} (-1)^{\nu/2} \frac{i!}{i_1!i_2!} \frac{(\nu-1)!!}{3(\nu)!!}(b)^3(x')^{2i_1} I_3(x')$$

$$+ \sum_{i_1,i_2} \sum_{j=2}^{\infty} (-1)^{\nu/2+j} \frac{i!}{i_1!i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j}(j!)^2(2j+1)}(b)^{2j+1}(x')^{2i_1} I_4(x') \tag{33}$$

where

$$I_1(x') = \int_{-a}^{a} (y')^{2n_2+\mu} \sqrt{b^2+x'^2+y'^2}\, dy'$$

$$= \Pi_\xi (a)^{\xi-2t_1+1}(b^2+x'^2)^{t_1-1}\sqrt{(a^2+b^2+x'^2)^3} + (-1)^{\xi/2} \frac{2(\xi-1)!!}{(\xi+2)!!}(b^2+x'^2)^{\xi/2}$$

$$\cdot \left(a\sqrt{a^2+b^2+x'^2} + (b^2+x'^2)\,\mathrm{sh}^{-1} \frac{a}{\sqrt{b^2+x'^2}}\right)$$

if μ is even (34a)

$= 0$ if μ is odd. (34b)

In the above expression

$$\Pi_\xi = \frac{2}{\nu+1} \sum_{t_1=1}^{\xi/2} (-1)^{t_1+1} \frac{(\xi+1)(\xi-1)\ldots(\xi+2t_1+3)}{(\xi+2)\xi\ldots(\xi-2t_1+4)} \tag{35}$$

$(\xi-1)!! = 1\cdot3\ldots(\xi-1)$ ξ is even (36a)

$(\xi+2)!! = 2\cdot4\ldots(\xi+2)$ ξ is even (36b)

$\xi = 2n_2+\mu$ (37)

The integral

$$I_2(x') = \int_{-a}^{a} (y')^{2i_2+\mu} \frac{dy'}{\sqrt{x'^2+y'^2}}$$

$$= \Pi_{\xi_1}(a)^{\xi_1-2t_3+1}(x'^2)^{t_3-1}\sqrt{a^2+x'^2} + (-1)^{\xi_1/2}\frac{2(\xi_1-1)!!}{(\xi_1)!!}(x'^2)^{\xi_1/2}\,\mathrm{sh}^{-1}\frac{a}{x'}$$

if μ is even (38a)

$= 0$ if μ is odd. (38b)

where Π_{ξ_1} is defined as the same as (32a), only if replace ν with ξ_1 and

$\xi_1 = 2i_2+\mu$ (39a)

$(\xi_1-1)!! = 1\cdot3\ldots(\xi_1-1)$ ξ_1 is even (39b)

$(\xi_1)!! = 2\cdot4\ldots(\xi_1)$ ξ_1 is even (39c)

The integral

$$I_3(x') = \int_{-a}^{a} \frac{(y')^{\xi_1}}{\sqrt{(x'^2+y'^2)^3}}\,dy'$$

$$= \Pi'_{\xi_1}(a)^{\xi_1-2t_4+1}(x'^2)^{t_4-1}/\sqrt{a^2+x'^2}$$

$$+ (-1)^{\frac{\xi_1-2}{2}}\frac{2(\xi_1-1)!!}{(\xi_1-2)!!}(x'^2)^{\frac{\xi_1}{2}-1}\left(\mathrm{sh}^{-1}\frac{a}{x'} - \frac{a}{\sqrt{a^2+x'^2}}\right)$$

if μ is even (40a)

$= 0$ if μ is odd. (40b)

where

$$\Pi'_{\xi_1} = \frac{2}{\xi_1+1} \sum_{t_4=1}^{\frac{\xi_1}{2}-1} (-1)^{t_4+1} \frac{(\xi_1+1)(\xi_1-1)\ldots(\xi_1-2t_4+3)}{(\xi_1-2)(\xi_1-4)\ldots(\xi_1-2t_4)} \tag{41}$$

The integral

$$I_4(x') = \int_{-a}^{a} \frac{(y')^{\xi_1}}{\sqrt{(x'^2+y'^2)^{2j+1}}} dy'$$

$$= \Pi''_{\xi_1} (a)^{\xi_1-2t_2+1} (x'^2)^{t_2-1} / \sqrt{(a^2+x'^2)^{2j-1}}$$

$$+ \sum_{t_o=1}^{j-1} (-1)^{\xi_1/2} \frac{2(\xi_1-1)!!(2)^{t_o}}{(\xi_1-2j)(\xi_1-2j-2)\ldots(2-2j)} \frac{(j-1)(j-2)\ldots(j-t_o)}{(2j-1)(2j-3)\ldots(2j-2t_o+1)}$$

$$\cdot (a)(x'^2)^{\frac{\xi_1}{2}-t_o-1} \frac{(a^2+x'^2)^{t_o}}{\sqrt{(a^2+x'^2)^{2j-1}}}$$

$$+ (-1)^{\xi_1/2} \frac{2(\xi_1-1)!!}{(\xi_1-2j)(\xi_1-2j-2)\ldots(2-2j)(2j-1)} (a)(x'^2)^{\frac{\xi_1}{2}-1} \cdot \frac{1}{\sqrt{(a^2+x'^2)^{2j-1}}}$$

if μ is even (42a)

$= 0$ if μ is odd (42b)

where

$$\Pi''_{\xi_1} = \frac{2}{\xi_1+1} \sum_{t_o=1}^{\xi_1/2} (-1)^{t_2+1} \frac{(\xi_1+1)(\xi_1-1)\ldots(\xi_1-2t_2+3)}{(\xi_1-2j)(\xi_1-2j-2)\ldots(\xi_1-2j-2t_2+2)} \tag{43}$$

(iii)

$$\iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{\sqrt{x'^2+y'^2+z'^2}}$$

$$= \Pi_{\nu}\Pi_{\xi} \sum_{n_1,n_2} (-1)^{\nu/2} \frac{n!}{n_1!n_2!}(a)^{\xi-2t_1+1}\cdot(b)^{\nu-2t+1}\cdot I_1$$

$$+ (-1)^{\xi/2}\Pi_{\nu} \frac{2(\xi-1)!!}{(\xi+2)!!}(b)^{\nu-2t+1}\cdot I_2$$

$$+ \Pi_{\xi_1} \sum_{i_1,i_2} (-1)^{\nu/2} \frac{i!}{i_1!i_2!} \frac{2(\nu-1)!!}{(\nu)!!}(a)^{\xi_1-2t_2+1}\cdot(b)\cdot I_3$$

$$+ \sum_{i_1,i_2} (-1)^{\frac{\nu+\xi_1}{2}} \frac{i!}{i_1!i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{2(\xi_1-1)!!}{(\xi_1)!!}(b)\cdot I_4$$

$$- \Pi'_{\xi_1} \sum_{i_1,i_2} (-1)^{\nu/2} \frac{i!}{i_1!i_2!} \frac{(\nu-1)!!}{3(\nu)!!}(a)^{\xi_1-2t_4+1}(b)^3\cdot I_5$$

$$- \sum_{i_1,i_2} (-1)^{\frac{\xi_1+\nu-1}{2}} \frac{i!}{i_1!i_2!} \frac{(\nu-1)!!}{3(\nu)!!} \frac{2(\xi_1-1)!!}{(\xi_1-2)!!}(b)^3\cdot I_6$$

$$+ \Pi'_{\xi_1} \sum_{i_1,i_2} \sum_{j=2}^{\infty} (-1)^{j+\frac{\nu}{2}} \frac{i!}{i_1!i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j}(j!)^2(2j+1)}(a)^{\xi_1-2t_2+1}\cdot(b)^{2j+1}\cdot I_7$$

$$+ \sum_{i_1,i_2} \sum_{j=2} \sum_{t_o=1}^{j-1} (-1)^{j+\frac{\nu}{2}+\frac{\xi_1}{2}} \frac{i!}{i_1!i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{2(\xi_1-1)!!}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)}$$

$$\cdot \frac{(2j)!}{2^{2j}(j!)^2(2j+1)} \frac{(j-1)(j-2)\dots(j-t_o)}{(2j-1)(2j-3)\dots(2j-2t_o+1)}(a)(b)^{2j+1}\cdot I_8$$

$$+ \sum_{i_1,i_2} \sum_{j=2}^{\infty} (-1)^{j+\frac{\nu}{2}+\frac{\xi_1}{2}} \frac{i!}{i_1!i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j}(j!)^2(2j+1)}$$

$$\cdot \frac{2(\xi_1-1)!!}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)(2j-1)}(a)(b)^{2j+1}\cdot I_9 \qquad (44)$$

The integrals in (44) can be evaluated as in Appendix.

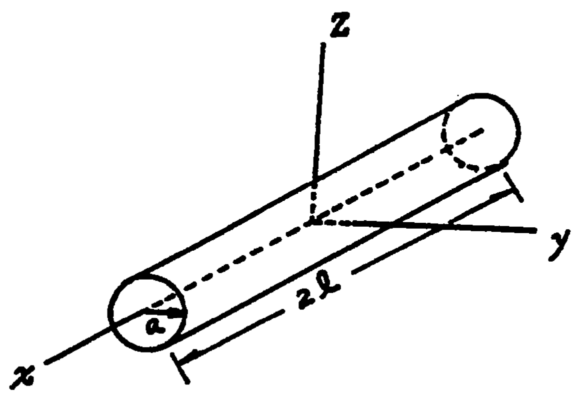

FIG. 1

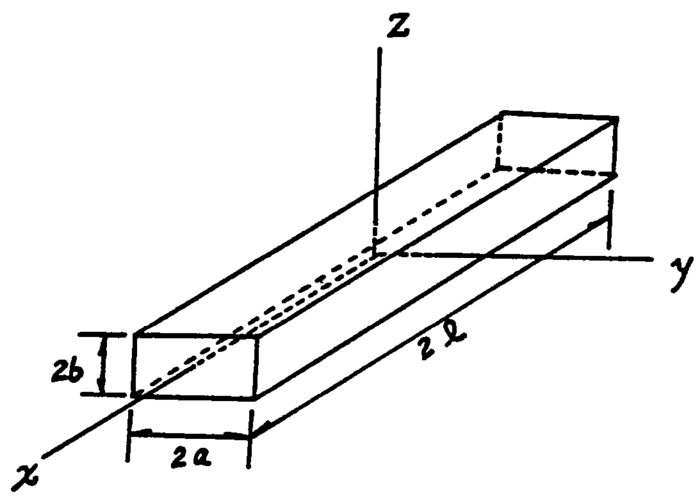

FIG. 2

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APPENDIX

The I-Integrals and π-Functions

(a) The I-integrals

(i)

$$I_1 = \int_{-\ell}^{\ell} (x')^{\lambda+2\eta}(b^2+x'^2)^{t_1-1}\sqrt{(a^2+b^2+x'^2)^3}\,dx'$$

$$= \pi_{\eta_\Gamma} \sum_{\Gamma_1,\Gamma_2} \frac{\Gamma!}{\Gamma_1!\Gamma_2!}(b)^{2\Gamma_1}\cdot(a^2+b^2)^{t_3-1}\cdot(\ell)^{\eta-2t_3+1}\sqrt{(a^2+b^2+\ \ ^2)^5}$$

$$+ \sum_{\Gamma_1,\Gamma_2}(-1)^{\eta/2}\frac{\Gamma!}{\Gamma_1!\Gamma_2!}\frac{8(\eta-1)!!}{(\eta+4)!!}(b)^{2\Gamma_1}\cdot(a^2+b^2)^{\eta/2}[\tfrac{1}{2}\ell\cdot\sqrt{(a^2+b^2+\ell^2)^3} +$$

$$+ \frac{3}{4}(a^2+b^2)(\ell\sqrt{a^2+b^2+\ell^2} + (a^2+b^2)\,sh^{-1}\frac{\ell}{\sqrt{a^2+b^2}})]$$

if λ is even

$= 0$ if λ is odd

in which the π-function π_η is defined in (b) of the Appendix.

(ii) $I_2 = I_2' + I_2''$

$$I_2' = \int_{-\ell}^{\ell} a(x')^{\lambda+2\eta_1}(b^2+x'^2)^{\xi/2}\sqrt{a^2+b^2+x'^2}\,dx'$$

$$= \Pi_{\eta_1}\sum_{\theta_1,\theta_2}\frac{\theta!}{\theta_1!\theta_2!}(a)(b)^{2\theta_1}\cdot(a^2+b^2)^{t_4-1}\cdot(\ell)^{\eta_1-2t_4+1}\cdot\sqrt{(a^2+b^2+\ell^2)^3}$$

$$+ \sum_{\theta_1,\theta_2}(-1)^{\eta_1/2}\frac{\theta!}{\theta_1!\theta_2!}\frac{2(\eta_1-1)!!}{(\eta_1+2)!!}(a)(b)^{2\theta_1}(a^2+b^2)^{\eta_1/2}$$

$$\cdot\ (\ell\sqrt{a^2+b^2+\ \ ^2} + (a^2+b^2)\,sh^{-1}\frac{\ell}{\sqrt{a^2+b^2}}$$

if λ is even

$= 0$ if λ is odd.

$$I_2'' = \int_{-\ell}^{\ell} (x')^{\lambda+2n_1} (b^2+x'^2)^{3\xi/2} sh^{-1} \frac{a}{\sqrt{b^2+x'^2}}\, dx'$$

$$= 2 \sum_{e_1,e_2} \frac{e!}{e_1!e_2!} \frac{a(b)^{2\theta_1}(\ell)^{\eta_3+1}}{\eta_3+1} - \frac{1}{3} \sum_{e_1!e_2!} \frac{e!}{e_1!e_2!} \Big(\sum_{t_6=1}^{\eta_3/2} (-1)^{t_6+1} \cdot$$

$$\cdot \frac{(b^2)^{t_6-1}(\ell)^{\eta_3-2t_6+1}}{\eta_3-2t_3+1} + (-1)^{\eta_3/2}(b)^{\eta_3-1} \cdot t_g^{-1} \frac{\ell}{b}) (a)^3 (b)^{2e_1} +$$

$$+ \sum_{e_1,e_2} \sum_{j_1=2}^{\infty} (-1)^{j_1} \frac{(2j_1)!}{2^{2j_1}(j_1)!(2j_1+1)} \frac{e!}{e_1!e_2!} (a)^{2j_1+1}$$

$$\cdot (\Pi_{\eta_3} \frac{(b^2)^{t_7-1}(\ell)^{\eta_3-2t_7+1}}{(b^2+\ell^2)^{j_1-1}} + (-1)^{\eta_3/2} \frac{(\eta_3-1)!!}{(\eta_3-2j_1+1)(\eta_3-2j-1)\ldots(3-2j_1)}$$

$$\cdot (b)^{\eta_3} (\Pi'_{\eta_3}(\ell) + (-1)^{j_1-1} \frac{2}{b} t_g^{-1} \frac{\ell}{b})](b)^{2e_1}$$

if λ is even

$= 0$ if λ is odd.

(iii) $$I_3 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+2t_3-2} \sqrt{a^2+x'^2}\, dx'$$

$$= \Pi_{\eta_2} (a^2)^{t_5-1} (\ell)^{\eta_2-2t_5+1} \sqrt{(a^2+\ell^2)^3} + (-1)^{\eta_2/2} \frac{2(\eta_2-1)!!}{(\eta_2+2)!!} (a^2)^{\eta_2/2}$$

$$\cdot (\ell\sqrt{a^2+\ell^2} + a^2 sh^{-1} \frac{\ell}{a})$$ if λ is even

$= 0$ if λ is odd.

(iv) $$I_4 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1} sh^{-1} \frac{a}{x'}\, dx'$$

$$= \frac{a}{\eta_9+1} [\Pi_{\eta_9} (a^2)^{t_{15}-1} (\ell)^{\eta_9-2t_{15}+1} \sqrt{a^2+\ell^2} + (-1)^{\eta_9/2} \frac{2(\eta_9-1)!!}{(\eta_9)!!} (a)^{\eta_9} sh^{-1} \frac{\ell}{a}]$$

if λ is even

$= 0$ if λ is odd.

(v) $$I_5 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+2t_4-2} \frac{dx'}{\sqrt{a^2+x'^2}}$$

$$= \Pi_{\eta_8} (a)^{t_{14}-1} \cdot (\ell)^{\eta_8-2t_{14}-1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_8/2} \frac{2(\eta_8-1)!!}{(\eta_8)!!} (a^2)^{\eta_8/2} \operatorname{sh}^{-1} \frac{\ell}{a}$$

if λ is even

$= 0$ if λ is odd.

(vi)

$$I_6 = I_6' - I_6''$$

$$I_6' = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2} \operatorname{sh}^{-1} \frac{a}{x'} \, dx'$$

$$= \frac{a}{\eta_{10}+1} [\Pi_{\eta_{10}} (a^2)^{t_{16}-1} \cdot (\ell)^{\eta_{16}-2t_{16}+1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_{10}/2} \frac{2(\eta_{10}-1)!!}{(\eta_{10})!!} \cdot$$

$$\cdot (a)^{\eta_{10}} \operatorname{sh}^{-1} \frac{\ell}{a}]$$ if λ is even

$= 0$ if λ is odd.

$$I_6'' = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2} \frac{a}{\sqrt{a^2+x'^2}} \, dx'$$

$$= \Pi_{\eta_5} (a)^{t_9} \cdot (\ell)^{\eta_5-2t_9-1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_5/2} \frac{2(\eta_5-1)!!}{(\eta_5)!!} (a)^{\eta_5/2} \operatorname{sh}^{-1} \frac{\ell}{a}$$

if λ is even

$= 0$ if λ is odd.

(vii) $$I_7 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+2t_2-2} \frac{dx'}{\sqrt{(a^2+x'^2)^{2j-1}}}$$

$$= \Pi_{\eta_6} (a^2)^{t_{10}-1} \frac{(\ell)^{\eta_6-2t_{10}+1}}{\sqrt{(a^2+\ell^2)^{2j-3}}} + (-1)^{\eta_6/2} \frac{2(\eta_6-1)!!}{(\eta_6-2j-2)(\eta_6-2j-4)\dots(-2j)}$$

$$\frac{a^{\eta_6}\ell}{(2j-3)a^2\sqrt{(a^2+\ell^2)^{2j-3}}}\left(1+\sum_{k_{11}=1}^{j-2}\frac{8^{k_{11}}(j-2)(j-3)\ldots(j-k_{11}-1)}{(2j-5)(2j-7)\ldots(2j-2k_{11}-3)}\frac{(a^2+\ {}^2)^{k_{11}}}{(4a^2)^{k_{11}}}\right)$$

if λ is even

$= 0$ if λ is odd.

(viii)
$$I_8 = \int_{-\ell}^{\ell}(x')^{\lambda+2i_1+\xi_1-2t_o-2}\frac{(a^2+x'^2)^{t_o}dx'}{\sqrt{(a^2+x'^2)^{2j-1}}}$$

$$= \Pi_{\eta_7}\sum_{\phi_1,\phi_2}\frac{\phi!}{\phi_1!\phi_2!}\frac{(a^2)^{t_{12}-1+\phi_1}(\ell)^{\eta_7-2t_{12}+1}}{\sqrt{(a^2+\ell^2)^{2j-3}}} + \sum_{\phi_1,\phi_2}\frac{\phi!}{\phi_1!\phi_2!}$$

$$\cdot\ (-1)^{\eta_7/2}\frac{2(\eta_7-1)!!}{(\eta_7-2j-2)(\eta_7-2j-4)\ldots(-2j)}$$

$$\frac{(a^2)^{\eta_7/2+\phi_1}\ell}{(2j-3)a^2\sqrt{(a^2+\ell^2)^{2j-3}}}\left(1+\sum_{k_{13}=1}^{j-2}\frac{8^{k_{13}}(j-2)(j-3)\ldots(j-k_{13}-1)(a^2+\ell^2)^{k_{13}}}{(2j-5)(2j-7)\ldots(2j-2k_{13}-3)(4a^2)^{k_{13}}}\right.$$

if λ is even

$= 0$ if λ is odd.

(ix)
$$I_9 = \int_{-\ell}^{\ell}(x')^{\lambda+2i_1+\xi_1-2}\frac{dx'}{\sqrt{(a^2+x'^2)^{2j-1}}}$$

$$= \Pi_{\eta_8}\frac{(a^2)^{t_{12}-1}(\ell)^{\eta_8-2t_{12}+1}}{\sqrt{(a^2+\ell^2)^{2j-1}}} + (-1)^{\eta_8/2}\frac{2(\eta_8-1)!!}{(\eta_8-2j-2)(\eta_8-2j-4)\ldots(-2j)}$$

$$\frac{a^{\eta_8}\ell}{(2j-3)a^2\sqrt{(a^2+\ell^2)^{2j-3}}}\left(1+\sum_{k_{12}=1}^{j-2}\frac{8^{k_{12}}(j-2)(j-3)\ldots(j-k_{12}-1)(a^2+\ell^2)^{k_{12}}}{(2j-5)(2j-7)\ldots(2j-2k_{12}-3)(4a^2)^{k_{12}}}\right)$$

if λ is even

$= 0$ if λ is odd.

(b) The π-Functions

The π-functions in the I-integrals listed in (a) of this Appendix are defined as follows:

(i)
$$\Pi_{\eta} = \frac{2}{\eta+1} \sum_{t_3=1}^{\eta/2} (-1) \frac{(\eta+1)(\eta-1)\ldots(\eta-2t_3+3)}{(\eta+4)(\eta+2)\ldots(\eta-2t_3+6)}$$

$\eta = \lambda + 2n_2 + 2\Gamma_2$ (is even)

$\Gamma_1 + \Gamma_2 = \Gamma = t_1-1$

(ii) Π_{η_1} as the same as (35), if replace ξ with η_1

$\eta_1 = \lambda + 2n_1 + 2\theta_2$ (is even)

$\theta_1 + \theta_2 = \theta = \xi/2$

(iii)
$$\Pi_{\eta_3} = \frac{2}{\eta_3+1} \sum_{t_7=1}^{\eta_7/2} (-1)^{t_7+1} \frac{(\eta_3+1)(\eta_3-1)\ldots(\eta_3-2t_7+3)}{(\eta_3-2j_1+1)(\eta_3-2j_1-1)\ldots(\eta_3-2j_1-2t_7+3)}$$

$$\Pi'_{\eta_3} = \frac{4}{3-2(j_1+1)} \sum_{t_8=1}^{j_1-1} (-1)^{t_8+1} \cdot (2)^{t_8-1} \cdot \frac{[3-2(j_1+1)][3-2j_1]\ldots[3-2(j_1-t_8+2)]}{(j_1-1)(j_1-2)\ldots(j_1-t_8)}$$

$\eta_3 = \lambda + 2n_1 + 2e_2$ (is even)

$e_1 + e_2 = e = \frac{3}{2}\xi$

(iv) Π_{η_2} as the same as (35), if replace ξ with η_2

$\eta_2 = \lambda + 2i_1 + 2t_3 - 2$ (is even)

(v) Π_{η_9} as the same as (32a), if replace ν with η_9

$\eta_9 = \lambda + 2i_1 + \xi_1$ (is even)

(vi) Π_{η_8} as the same as (32a), if replace ν with η_8

$$\eta_8 = \lambda + 2i_1 + 2t_4 - 2 \quad \text{(is even)}$$

(vii) $\Pi_{\eta_{10}}$ as the same as (32a), if replace ν with η_{10}

$$\eta_{10} = \lambda + 2i_1 + \xi_1 - 2 \quad \text{(is even)}$$

(viii) Π_{η_5} as the same as (32a), if replace ν with η_5

$$\eta_5 = \lambda + 2i_1 + \xi_1 - 2 \quad \text{(is even)}$$

(ix)
$$\Pi_{\eta_6} = \frac{2}{\eta_6+1} \sum_{t_{10}=1}^{\eta_6/2} (-1)^{t_{10}+1} \frac{(\eta_6+1)(\eta_6-1)\dots(\eta_6-2t_{10}+3)}{(\eta_6-2j-2)(\eta_6-2j-4)\dots(\eta_6-2j-2t_{10})}$$

$$\eta_6 = \lambda + 2i_1 + 2t_2 - 2 \quad \text{(is even)}$$

(x) Π_{η_7} as the same as Π_{η_6}, if replace η_6 with η_7

$$\eta_7 = \lambda + 2i_1 + \xi_1 + 2\phi_2 - 2t_o - 2 \quad \text{(is even)}$$

$$\phi_1 + \phi_2 = \phi = t_o$$

(xi) Π_{η_8} as the same as Π_{η_6}, if replace η_6 with η_8

$$\eta_8 = \lambda + 2i_1 + \xi_1 - 2 \quad \text{(is even)}$$

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16. Abstract
Problems of wave phenomena in fields of acoustics, electromagnetics and elasticity are often reduced to an integration of the inhomogenous Helmholtz equation. Results are presented for volume integrals associated with the Helmholtz operator, $\nabla^2 + \alpha^2$, for the cases of an ellipsoidal region, a finite cylindrical region, and a region of rectangular parallelepiped. By using appropriate Taylor series expansions and multinomial theorem, these volume integrals are obtained in series form for regions $r > r'$ and $r < r'$, where r and r' are distances from the origin to the point of observation and source, respectively. Derivatives of these integrals are easily evaluated. When the wave number approaches zero, the results reduce directly to the potentials of variable densities.

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